ON THE LINEAR STABILITY OF KÄHLER-RICCI SOLITONS

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ABSTRACT. We show that Kähler-Ricci solitons with $\dim H^{(1,1)}(M) \geq 2$ are linearly unstable. This extends the results of Cao-Hamilton-Ilmanen in the Kähler-Einstein case.

1. INTRODUCTION

A Ricci soliton is a complete Riemannian metric g, a vector field X and a real number c satisfying

$$Ric(g) + \mathcal{L}_X g - cg = 0.$$

The soliton is called shrinking, steady or expanding if $c > 0, c = 0$ or $c < 0$ respectively. If $X = \nabla f$ for a smooth function f the soliton is said to be *a gradient Ricci soliton* with potential function f. If $X = 0$ we recover the definition of an Einstein metric. Solitons are important in the theory of the Ricci flow as they occur as fixed points of the flow, up to the action of the diffeomorphism group. Perelman's groundbreaking observation [9] was that solitons are the critical points of a certain functional (the ν-functional which we define below). An interesting question is whether the functional is locally maximised at these points (i.e. linearly stable) as this affects the behaviour of the Ricci-flow nearby to a soliton. It is this question that we take up in this article for a special class of gradient shrinking solitons, namely *Kähler-Ricci solitons*. These are solitons on Fano manifolds which have the additional property that g is a Kähler metric and the vector field ∇f is holomorphic.

The only known examples of compact Ricci solitons which are not products or Einstein are the Koiso-Cao soliton [7], [1], a $U(2)$-invariant soliton explicitly constructed on $\mathbb{CP}^2 \sharp - \mathbb{CP}^2$ and a class of toric-Kähler metrics constructed on Fano manifolds by Wang and Zhu [12].

The precise theorem we wish to prove is the following:

Theorem 1.1. *If (M, g, J) is a Kähler-Ricci Soliton with $\dim H^{(1,1)}(M) \geq 2$, then the soliton is linearly unstable.*

Corollary 1.2. The Koiso-Cao soliton on $\mathbb{CP}^2 \sharp - \mathbb{CP}^2$ and the Wang-Zhu soliton on $\mathbb{CP}^2 \sharp - 2\mathbb{CP}^2$ are both linearly unstable.

This result generalises the result of Cao-Hamilton-Ilmanen who gave a simple argument in [4] for Kähler-Einstein metrics satisfying $\dim H^{(1,1)}(M) \geq 2$. The generalisation to Kälher-Ricci solitons is also suggested in their paper. We begin by recalling some facts about the ν-functional, then proceed to compute the second variation of ν and define linear stability precisely. We finally show how one can construct unstable variations in a manner similar to [4].

The question of linear stability for Kähler-Ricci solitons has also been considered by Tian and Zhu [10]. They prove that if one considers variations in Kahler metrics in the fixed class $c_1(M)$ then the ν-energy is maximised at a Kahler-Ricci soliton. There is also the recent work of Cao and Meng-Zhu [5] who also give a detailed calculation of the second variation of ν.

2. THE PERELMAN ν-FUNCTIONAL

Throughout this paper (M, g) will denote a closed Riemannian manifold. Most of the material here is contained in some form in [11] and [2] and of course, was originally taken from [9]. Perelman defined the following functional on triples (g, f, τ), where g is a Riemannian metric, f a smooth function and $\tau > 0$ a constant.

Definition 2.1. Let f be a smooth function on M and $\tau > 0$ a real number. Then the W-functional is given by

$$W(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g,$$

where R is the scalar curvature of the metric g.

Some authors [10] absorb the constant τ into the other two terms and it is not hard to see that $\mathcal{W}(g, f, \tau) = \mathcal{W}(\tau^{-1}g, f, 1)$. The $\mathcal{W}$-functional is also invariant under diffeomorphisms i.e. $\mathcal{W}(g, f, \tau) = \mathcal{W}(\phi^*g, \phi^*f, \tau)$ for any diffeomorphism $\phi : M \to M$. Fix a compatibility condition for the triple (g, f, τ) by requiring

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} dV_g = 1.$$

This leads to the definition of the ν-functional:

Definition 2.2. Let f be a smooth function on M and $\tau > 0$ a real number. Then the W-functional is given by

$$\nu(g) = \inf\{\mathcal{W}(g, f, \tau) : (g, f, \tau) \text{ is compatible}\}.$$

We will not comment on the existence theory except to say that for fixed g there exists a $\tau > 0$ and smooth f that attain the infimum in the above definition. The pair (f, τ) satisfy the equations

$$\tau(-2\Delta f + |\nabla f|^2 - R) - f + n + \nu = 0, \text{ and } (4\pi\tau)^{-\frac{n}{2}} \int_M f e^{-f} dV_g = \frac{n}{2} + \nu.$$

The first important result about the ν-function is the following:

Theorem 2.3 (Perelman [9]). *Let $g(t)$ be a family of Riemannian metrics on M evolving via the Ricci-flow. Then $\nu(g(t))$ is montone increasing, unless g is a Ricci-soliton in which case ν is stationary.*

We also record the first variation of ν which makes it clear that stationary points of the functional are shrinking gradient Ricci solitons.

Theorem 2.4 (Perelman [9])**.** *The first variation of ν in the direction h, $\mathcal{D}_g\nu(h)$ is given by*

$$-(4\pi\tau)^{-\frac{n}{2}}\int_M \langle \tau(Ric(g) + Hess(f)) - \frac{1}{2}g, h\rangle e^{-f}dV_g.$$

3. The second variation of ν

The aim of this section give a self-contained proof of the second variation formula of ν at a Ricci soliton with potential function f. This is not new and has been known to experts for some time but we include it here for completeness. We work with the scaled L^2-inner product on tensors

$$\langle\cdot,\cdot\rangle_f = \int_M \langle\cdot,\cdot\rangle e^{-f}dV_g.$$

This inner product is adapted to the following operators

$$div_f(h) = e^f div(e^{-f}h) \text{ and } \Delta_f(h) = \Delta(h) - (\nabla_{\nabla f}h),$$

in the sense that

$$\int_M div_f(\alpha)e^{-f}dV_g = 0 \text{ and } \int_M \Delta_f(F)e^{-f}dV_g = 0$$

for any one-form α and function F. Obviously these reduce to the usual divergence and Laplacian when the metric is Einstein. The operator Δ_f is often referred to as the Bakry-Émery Laplacian. The sign convention for the curvature tensor we adopt is

$$R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$$

and $Rm(X,Y,W,Z) = g(R(X,Y)W,Z)$. In index notation we have $R(\partial_i,\partial_j,\partial_k,\partial_l) = Rm_{ijkl}$. For $h \in s^2(TM^*)$, define the symmetric curvature operator $Rm(h,\cdot) \in s^2(TM^*)$ by

$$Rm(h,\cdot)_{ij} = R_{kilj}h^{kl}.$$

We will also need the curvature operator on 2-forms, usually denoted by $\mathcal{R}$

$$\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M), \quad \mathcal{R}(\sigma)_{ij} = Rm_{ijkl}\sigma^{kl}.$$

The convention for divergence we adopt is $div(h) = tr_{12}(\nabla h)$. The reader should note that this definition is the opposite sign to the divergence operator considered in [11]. When restricted to forms we also have the codifferential δ which, with this convention, satisfies $\delta(\sigma) = -div(\sigma)$. If we denote by div_f^* the adjoint to div_f with respect to the scaled L^2-inner product $\langle\cdot,\cdot\rangle_f$, then, as remarked in [3], $div_f^* = div^*$. Here div^* is the adjoint to div with respect to the usual inner product on tensors.

Theorem 3.1 (Cao-Hamilton-Ilmanen)**.** *Let g be an Ricci-Soliton with potential function f satisfying $Ric(g) + Hess(f) = \frac{1}{2\tau}g$. For $h \in s^2(TM^*)$, consider variations $g(s) = g + sh$. Then the second variation of the ν-energy is*

$$\frac{d^2}{ds^2}\nu(g(s))\Big|_{s=0} = \frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\int_M \langle h, Nh\rangle e^{-f}dV_g,$$

where N is given by

$$N(h) = \frac{1}{2}\Delta_f(h) + Rm(h,\cdot) + div_f^*div_f(h) + \frac{1}{2}Hess(v_h) - C(h,g)Ric.$$

Here v_h is the solution of

(3.1) $$\Delta_f(v_h) + \frac{1}{2\tau}v_h = div_f div_f(h)$$

and $C(h,g)$ is a constant depending upon h and g.

Remark: This theorem was first stated in [3] but with an error in the term $C(h,g)$. This has subsequently been corrected in the recent work of Cao and Meng-Zhu [5]. They find that

$$C(h,g) = \frac{\int_M \langle Ric, h\rangle e^{-f} dV_g}{\int_M Re^{-f} dV_g}$$

where R is the scalar curvature of g.

Definition 3.2. A soliton is linearly stable if the operator N is non-positive definite and linearly unstable otherwise.

We now collect some formulae for how various geometric quantities vary through a family of Riemannian metrics $g(t)$ evolving via $\frac{\partial g}{\partial t} = h$ where $h \in s^2(TM^*)$. Much of what is required is explained extremely well in [11] and so many proofs are omitted.

Lemma 3.3 (Prop 2.3.7 and 2.3.9 in [11]). *Let $g(s) = g + sh$ be a family of Riemannian metrics on M. Then*

$$\frac{d}{ds}Ric(g(s))\Big|_{s=0} = \frac{1}{2}\nabla^*\nabla(h) - Rm(h,\cdot) + \frac{1}{2}(Ric\cdot h + h\cdot Ric) - div^* div(h) - \frac{1}{2}Hess(tr(h)).$$

Let $R(s) = tr_{g(s)}Ric(g(s))$ be the scalar curvature of $g(s)$ then

$$\frac{d}{ds}R(s)\Big|_{s=0} = -\langle h, Ric\rangle + divdiv(h) - \Delta(tr(h)).$$

It is necessary to know how geometric quantities associated to functions $f : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ vary. The convention we adopt is that $\Delta f = tr(Hess(f)) = -\delta df$, so the Laplacian has negative eigenvalues.

Lemma 3.4 (Prop 2.3.10 in [11]). *Let $g(s) = g + sh$ be a family of Riemannian metrics on M and let $f(s,x) : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ be a family of smooth functions. Then*

$$\frac{d}{ds}\Delta(f)\Big|_{s=0} = \Delta_g(\dot{f}) - \frac{1}{2}\langle d(tr(h)), df\rangle - \langle h, Hess(f)\rangle - \langle div(h), df\rangle$$

$$\frac{d}{ds}|\nabla f|^2\Big|_{s=0} = -h(\nabla f, \nabla f) + 2\langle \nabla f, \nabla \dot{f}\rangle$$

The final object whose variation must be computed is the Hessian of the functions $f(s,x)$.

Lemma 3.5. *Let $g(s) = g + sh$ be a family of Riemannian metrics on M and let $f(s,x) : (-\epsilon, \epsilon) \times M \to \mathbb{R}$ be a family of smooth functions. Then*

$$\frac{d}{ds}Hess(f)\Big|_{s=0} = Hess(\dot{f}) + \frac{1}{2}(Hess(f)\cdot h + h\cdot Hess(f)) + \frac{1}{2}(\nabla_{\nabla f}h) + div^*(h(\nabla f, \cdot)).$$

Proof. The Hessian of a function f is given by $Hess(X, Y) = g(\nabla_X \nabla f, Y)$. Hence

$$\frac{d}{ds} Hess(X, Y) = h(\nabla_X \nabla f, Y) + g(\frac{d}{ds}(\nabla_X \nabla f), Y)$$

$$= h(\nabla_X \nabla f, Y) + g(\frac{d}{ds}(\nabla_X) \nabla f, Y) + g(\nabla_X \frac{d}{ds} \nabla f, Y).$$

By Proposition 2.3.1 in [11]

$$g(\frac{d}{ds}(\nabla_X) \nabla f, Y) = \frac{1}{2}[(\nabla_{\nabla f} h)(X, Y) + (\nabla_X h)(Y, \nabla f) - (\nabla_Y h)(X, \nabla f)].$$

We also have

$$g(\nabla_X \frac{d}{ds} \nabla f, Y) + h(\nabla_X \nabla f, Y) = g(\nabla_X \nabla \dot{f}, Y) - (\nabla_X h)(Y, \nabla f)$$

$$= Hess(\dot{f})(X, Y) - (\nabla_X h)(Y, \nabla f).$$

Hence the variation is given by

$$Hess(\dot{f})(X, Y) + \frac{1}{2}(\nabla_{\nabla f} h)(X, Y) + \frac{1}{2}[h(\nabla_X \nabla f, Y) + h(\nabla_Y \nabla f, X)]$$

$$- \frac{1}{2}[(\nabla_X h(\nabla f, \cdot))(Y) + (\nabla_Y h(\nabla f, \cdot))(X)],$$

and the result follows. $\square$

Lemma 3.6. *For $h \in s^2(TM^*)$, then*

$$div_f div_f(h) = divdiv(h) + h(\nabla f, \nabla f) - 2\langle df, div(h)\rangle - \langle h, Hess(f)\rangle.$$

Proof. From the definition of div_f we have $div_f(h)(\cdot) = div(h)(\cdot) - h(\nabla f, \cdot)$ so we need to compute

$$div_f(div(h)(\cdot) - h(\nabla f, \cdot)) = e^f div(e^{-f}(div(h)(\cdot) - h(\nabla f, \cdot))$$

$$= divdiv(h) - div(h)(\nabla f) - div(h(\nabla f, \cdot)) + h(\nabla f, \nabla f).$$

The term $div(h(\nabla f, \cdot)) = div(h)(\nabla f) + \langle h, Hess(f)\rangle$, and so

$$div_f div_f(h) = divdiv(h) + h(\nabla f, \nabla f) - 2\langle df, div(h)\rangle - \langle h, Hess(f)\rangle.$$

$\square$

If the soliton varies by $\delta g = h$, then this induces a variation in the pair (f, τ) which is denoted $(\delta f, \delta \tau)$.

Lemma 3.7. *Let (g, f, τ) describe a a Ricci-Soliton and consider a variation $g(s) = g + sh$ inducing variations $(\delta f, \delta \tau)$ in f and τ. If $v_h = (tr(h) - 2\delta f - \frac{2\delta \tau}{\tau}(f - \nu))$, then v_h satisfies*

$$\Delta_f(v_h) + \frac{v_h}{2\tau} = div_f div_f(h)$$

and

$$\int_M (-\frac{n\delta\tau}{2\tau} f + \delta f(1 - f) + f\frac{1}{2} tr(h)) e^{-f} dV_g = 0.$$

Proof. Consider the variation in the equation

$$\tau(-2\Delta f + |\nabla f|^2 - R) - f + n + \nu = 0,$$

which yields

$$\delta\tau(-2\Delta f + |\nabla f|^2 - R) + \tau(-2\delta(\Delta f) + \delta|\nabla f|^2 - \delta R) - \delta f + \delta\nu = 0.$$

As we are at a soliton, $\delta\nu = 0$ and $R = \frac{n}{2\tau} - \Delta f$. Hence

$$-2\Delta f + |\nabla f|^2 - R = -\Delta f + |\nabla f|^2 - \frac{n}{2\tau} = -\Delta_f f - \frac{n}{2\tau}.$$

Lemmas 3.3 and 3.4 imply that

$$\delta(-2\Delta f) = \Delta(-2\delta f) + \langle \nabla tr(h), \nabla f \rangle + 2\langle h, Hess(f) \rangle + 2\langle div(h), df \rangle$$

and

$$-\delta R = \langle h, Ric \rangle - divdiv(h) + \Delta(tr(h)) = -\langle h, Hess \rangle + \frac{1}{2\tau}tr(h) - divdiv(h) + \Delta(tr(h)).$$

This yields

$$\frac{\delta\tau}{\tau}(-2\Delta f + |\nabla f|^2 - R) + \Delta(tr(h) - 2\delta f) + \langle \nabla(tr(h) - 2\delta f), \nabla f \rangle + \frac{1}{2\tau}(tr(h) - 2\delta f)$$
$$= divdiv(h) + h(\nabla f, \nabla f) - \langle h, Hess \rangle - 2\langle div(h), df \rangle.$$

The left hand side is $div_f div_f(h)$, by Lemma 3.7, and the right hand side may be written as

$$\frac{\delta\tau}{\tau}(-2\Delta f + |\nabla f|^2 - R) + \Delta_f(tr(h) - 2\delta f) + \frac{1}{2\tau}(tr(h) - 2\delta f).$$

At a Ricci soliton, the scalar curvature R is given by $R = -\Delta f + \frac{n}{2\tau}$ so

$$\Delta_f(f) = \Delta(f) - |\nabla f|^2 = 2\Delta(f) - |\nabla f|^2 + R - \frac{n}{2\tau} = \frac{-f}{\tau} + \frac{n}{2\tau} + \frac{\nu}{\tau},$$

whence

$$\Delta_f f + \frac{f}{2\tau} = -\frac{f}{2\tau} + \frac{n}{2\tau} + \frac{\nu}{\tau}.$$

Setting $\tilde{f} = -\frac{2\delta\tau}{\tau}(f - \nu)$,

$$\Delta_f(\tilde{f}) + \frac{\tilde{f}}{2\tau} = \frac{\delta\tau}{\tau}(\frac{f}{\tau}) - \frac{2\delta\tau}{\tau}(\frac{n}{2\tau} + \frac{\nu}{\tau}) + \frac{\delta\tau\nu}{\tau^2} = \frac{\delta\tau}{\tau}(-2\Delta f + |\nabla f|^2 - R).$$

Now letting $v_h = tr(h) - 2\delta f - 2\frac{\delta\tau(f-\nu)}{\tau}$,

$$\Delta_f(v_h) + \frac{v_h}{2\tau} = div_f div_f(h).$$

The second equation is simply the variation in the equation

$$(4\pi\tau)^{\frac{-n}{2}}\int_M fe^{-f}dV_g = \frac{n}{2} + \nu.$$

$\square$

We proceed to the proof of Theorem 3.1:

Proof. The first variation is given by

$$\frac{d}{ds}\nu(g)|_{s=0} = (4\pi\tau)^{-\frac{n}{2}}\int_M -\langle h, \tau(Ric + Hess(f)) - \frac{1}{2}g\rangle e^{-f}dV_g$$

so it is sufficient to compute the variation in the term

$$\tau(Ric) + Hess(f)) - \frac{1}{2}g.$$

This is given by

$$\delta\tau(Ric(g) + Hess(f)) - \frac{1}{2}h + \tau(\delta Ric + \delta Hess(f)).$$

Using the previous results

$$\delta Ric = \frac{1}{2}\nabla^*\nabla(h) - Rm(h,\cdot) + \frac{1}{2}(Ric \cdot h + h \cdot Ric) - div^*div(h) - \frac{1}{2}Hess(tr(h)),$$

which, using the fact we are at a soliton, yields

$$\delta Ric = \frac{1}{2}\nabla^*\nabla(h) - Rm(h,\cdot) - \frac{1}{2}(Hess(f) \cdot h + h \cdot Hess(f)) + \frac{h}{2\tau}$$
$$- div^*div(h) - \frac{1}{2}Hess(tr(h)).$$

The variation in the Hessian is given by

$$Hess(\dot{f}) + \frac{1}{2}(Hess(f) \cdot h + h \cdot Hess(f)) + \frac{1}{2}(\nabla_{\nabla f}h) + div^*(h(\nabla f,\cdot)).$$

Putting this together the variation is given by

$$\delta\tau Ric(g) - \frac{\tau}{2}(\Delta(h) - (\nabla_{\nabla f}h)) - \tau Rm(h,\cdot) - \tau div^*(div(h) - h(\nabla f,\cdot))$$
$$+ \frac{\tau}{2}Hess(\frac{2\delta\tau}{\tau}f + 2\delta f - tr(h)),$$

which may be rewritten as

$$= \delta\tau Ric(g) + \tau(-\frac{1}{2}\Delta_f(h) - Rm(h,\cdot) - div_f^*div_f(h) - \frac{1}{2}Hess(v_h)).$$

The result follows on taking $\frac{\delta\tau}{\tau} = C(h,g)$. $\qquad\square$

Taking f constant recovers:

Corollary 3.8 (Einstein Case [4]). *Let g be an Einstein metric with $Ric(g) = \frac{1}{2\tau}g$. For $h \in s^2(TM^*)$, consider variations $g(s) = g + sh$. Then the second variation of the ν-energy at is*

$$\frac{d^2}{ds^2}\bigg|_{s=0} \nu(g(s)) = \frac{\tau}{vol(g)}\int_M \langle h, Nh\rangle dV_g,$$

where N is given by

$$N(h) = -\frac{1}{2}\nabla^*\nabla(h) + Rm(h,\cdot) + div^*div(h) + \frac{1}{2}Hess(v_h) - \frac{g}{2n\tau vol(g)}\int_M tr(h)dV_g.$$

Here v_h is the solution of

$$\Delta(v_h) + \frac{1}{2\tau}v_h = divdiv(h).$$

4. The instability of Kähler-Ricci solitons

We first establish the following simple Lemma comparing curvature operators on a Kahler manifold:

Lemma 4.1. *Let (M,g,J) be a Kahler manifold. If $\sigma \in \mathcal{A}^{(1,1)}(M)$, then $2Rm(\sigma_J,\cdot) - \mathcal{R}(\sigma)_J = 0$.*

Proof. We calculate using an adapted orthonormal basis $\{e_i, Je_i\}$. From the definition of $\mathcal{R}$, the Bianchi identity and the J-invariance of the curvature tensor;

$$
\begin{aligned}
\mathcal{R}(\sigma)(X, JY) &= \sum_{i,j} R(e_i, e_j, X, JY)\sigma(e_i, e_j) \\
&= -\sum_{i,j} \left(R(X, e_i, e_j, JY) + R(e_j, X, e_i, JY) \right) \sigma(e_i, e_j) \\
&= \sum_{i,j} \left(R(X, e_i, Je_j, Y) + R(e_j, X, Je_i, Y) \right) \sigma(e_i, e_j) \\
&= \sum_{i,j} -2R(e_i, X, Je_j, Y)\sigma(e_i, e_j) \\
&= \sum_{i,j} 2R(e_i, X, Je_j, Y)\sigma(e_i, J(Je_j)) \\
&= 2Rm(\sigma_J, \cdot)(X, Y).
\end{aligned}
$$

$\square$

The proof of the main result will be modeled on the Kahler-Einstein case which is outlined in [4].

Proposition 4.2. *Let (M, g, J) be a Kähler-Einstein manifold with $\dim H^{(1,1)}(M) \geq 2$. Then g is linearly unstable.*

Proof. Choose a trace-free harmonic $(1,1)$-form $\sigma \in \mathcal{H}^{(1,1)}(M)$ that induces a perturbation $\sigma_J = \sigma(\cdot, J\cdot)$. As the complex structure is parallel $div(\sigma_J) = 0$ and $(\nabla^*\nabla\sigma_J) = (\nabla^*\nabla\sigma)_J$. We can use a Weitzenbock formula on 2-forms to compare the rough Laplacian with the Hodge Laplacian $\Delta_H = -(d+\delta)^2$. The Weitzenbock formula for an Einstein metric with Einstein constant $\frac{1}{2\tau}$ is

$$
-\Delta_H = \nabla^*\nabla - \mathcal{R} + \frac{1}{\tau}id
$$

where $\mathcal{R}$ is the curvature operator for 2-forms. Our formula for the variation becomes

$$
\begin{aligned}
2N(\sigma_J) &= \Delta_H(\sigma)_J - \mathcal{R}(\sigma)_J + \frac{1}{\tau}\sigma_J + 2Rm(\sigma_J, \cdot) + Hess(v_{\sigma_J}) \\
&= -\mathcal{R}(\sigma)_J + \frac{1}{\tau}\sigma_J + 2Rm(\sigma_J, \cdot) + Hess(v_{\sigma_J}).
\end{aligned}
$$

It follows from Equation 3.1 that the function v_{σ_J} is an eigenfuction of the Laplacian. Recall that the first non-zero eigenvalue of the Laplacian on an Einstein manifold with Einstein constant $c > 0$ satisfies

$$
\lambda_1 \leq \frac{-n}{n-1}c
$$

by the famous Lichnerowicz bound. Hence $v_{\sigma_J} = 0$. From Lemma 4.1, $2Rm(\sigma_J, \cdot) - \mathcal{R}(\sigma)_J = 0$ so we have exhibited an eigentensor for N with eigenvalue $\frac{1}{2\tau}$. Hence metrics of this type are unstable. $\square$

Considering the divergence operator $\delta : \Omega(M)^k \to \Omega(M)^{k-1}$ as the adjoint of the exterior derivative on forms then, with our conventions, $div(\sigma) = -\delta(\sigma)$ for any k-form σ. Set

$$
\delta_f(\sigma) = e^f \delta(e^{-f}\sigma) = \delta(\sigma) + \iota_{\nabla f}\sigma \text{ and } \Delta_{f,H}(\sigma) = -(d\delta_f + \delta_f d)(\sigma).
$$

We will refer to $\Delta_{f,H}$ as the *twisted Laplacian*. Forms σ satisfying $\Delta_{f,H}(\sigma) = 0$ are said to be *twisted harmonic forms*. Modifications of the Hodge Laplacian similar to this are also found in the seminal work of Witten [13] on the Morse inequalities. The following Lemma gives some crucial properties of this Laplacian:

Lemma 4.3. *The operator $\Delta_{f,H}$ has the following properties:*
  (1) $\Delta_{f,H} = \Delta_H - \mathcal{L}_{\nabla f}$.
  (2) $\Delta_{f,H}$ *preserves the decomposition into $(p,q) - forms$.*
  (3) $\Delta_{f,H}$ *satisfies a Weitzenbock identity for 2-forms σ:*

$$\Delta_f(\sigma) - \Delta_{f,H}(\sigma) = -\mathcal{R}(\sigma) + \frac{1}{\tau}\sigma.$$

Proof. From the definition we have

$$\Delta_{f,H}(\sigma) = \Delta_H(\sigma) - (d \circ \iota_{\nabla f} + \iota_{\nabla f} \circ d)\sigma.$$

Using Cartan's magic formula this yields

$$\Delta_{f,H}(\sigma) = \Delta_H(\sigma) - \mathcal{L}_{\nabla f}(\sigma).$$

The second claim follows from the fact that ∇f is a holomorphic vector field. For the Weitzenbock formula recall that for 2 forms

$$-\nabla^*\nabla(\sigma) - \Delta_H(\sigma) = -\mathcal{R}(\sigma) + Ric \cdot \sigma + \sigma \cdot Ric$$

$$= -\mathcal{R}(\sigma) - (Hess(f) \cdot \sigma + \sigma \cdot Hess(f)) + \frac{1}{\tau}\sigma,$$

where $\mathcal{R}$ is the curvature operator for 2-forms. A simple computation yields

$$\mathcal{L}_{\nabla f}(\sigma) - (\nabla_{\nabla f})(\sigma) = (Hess(f) \cdot \sigma + \sigma \cdot Hess(f)).$$

Hence

$$(\Delta_f - \Delta_{f,H})(\sigma) = -\mathcal{R}(\sigma) + \frac{1}{\tau}\sigma.$$

$\square$

To conclude the proof of Theorem 1.1 is presented:

Proof. The first step is to choose a twisted harmonic form $\sigma \in H^{(1,1)}(M)$. If we denote the space of twisted harmonic forms by $\mathcal{H}_f$ then we have the natural map

$$\pi : \mathcal{H}_f \to H^2(M) \text{ given by } \pi(\sigma) = [\sigma].$$

The proof that this map is an isomorphism follows that of the usual Hodge theorem except that the 'energy' in our case is given by

$$\int_M \|\sigma\|^2 e^{-f} dV_g.$$

The twisted Laplacian also preserves the (p,q)-decomposition of forms so we get the required isomorphism of vector spaces

$$\mathcal{H}_f^{(1,1)} \cong H^{(1,1)}(M).$$

The hypothesis on the dimension of $H^{(1,1)}(M)$ means that there exist $\theta_1, \theta_2 \in \mathcal{H}_f^{(1,1)}$ and $\lambda, \mu \in \mathbb{R}$, not both zero, satisfying

$$\langle \lambda\theta_1 + \mu\theta_2, \rho \rangle_f = \int_M \langle \lambda\theta_1 + \mu\theta_2, \rho \rangle e^{-f} dV_g = 0,$$

where ρ is the Ricci form. Choosing $\sigma = \lambda\theta_1 + \mu\theta_2$ and computing gives

$$\langle N(\sigma_J), \sigma_J \rangle_f = \frac{1}{2\tau}\|\sigma_J\|^2 > 0,$$

and the theorem follows. $\square$

Corollary 1.2 follows immediately from this theorem.

Remark: In the recent work [5] the authors consider the operator

$$\hat{\mathcal{L}} = \frac{1}{2}\Delta_f + Rm(h, \cdot).$$

They prove that the Ricci form ρ is twisted harmonic and that it satisfies $\hat{\mathcal{L}}(Ric) = \frac{1}{2\tau}Ric$. Our proof here can be phrased as showing that the multiplicity of this eigenspace is at least $\dim H^{(1,1)}(M)$. One may then apply their Proposition 3.1 to conclude that Kähler-Ricci solitons are linearly unstable.

It would be most interesting to determine whether the perturbations σ_J are in fact eigentensors of the operator N. It is clear that if $v_{\sigma_J} = 0$ then σ_J is an eigentensor. We observe that the recent work of Futaki-Sano [6] and Ma [8] gives a spectral gap for the Bakry-Émery Laplacian when the Bakry-Émery Ricci curvature is bounded below. They show:

Theorem 4.4 (Ma, Futaki-Sano). *Let ϕ be a smooth function on M and suppose that the Bakry-Émery Ricci curvature, defined as*

$$Ric_\phi = Ric(g) + Hess(\phi),$$

satisfies the bound

$$Ric_\phi \geq c,$$

for some number $c > 0$. Then the first non-zero eigenvalue λ_1 of the Bakry-Émery Laplacian $\Delta_\phi := \Delta - \nabla_{\nabla\phi}$ satisfies

$$\lambda_1 \leq -c.$$

Clearly this bound is weaker that the Lichnerowicz bound in the Einstein case. Hence one may not deduce that $v_{\sigma_J} = 0$ for the unstable variations of Theorem 1.1. However, if $v_{\sigma_J} \neq 0$ such a soliton would prove that the above estimate is in some sense sharp. We suspect however that $v_{\sigma_j} = 0$ and hence that these variations are eigentensors of N. Implicit in the calculation of the second variation formula is the fact that the potential function f (once normalised) is an eigenfunction of Δ_f with eigenvalue $-\frac{1}{\tau}$ (Futaki and Sano also prove this). It would be interesting to know if Δ_f has any eigenvalues in the interval $(-\frac{1}{\tau}, -\frac{1}{2\tau}]$ and the authors are currently numerically investigating this problem for the Koiso-Cao and Wang-Zhu solitons.

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